

# ON 6-CANONICAL MAP OF IRREGULAR THREEFOLDS OF GENERAL TYPE

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ABSTRACT. We prove that, for any nonsingular projective irregular 3-fold of general type, the 6-canonical map is birational onto its image.

## 1. Introduction

Given a nonsingular projective variety  $V$  of general type, by definition, the pluricanonical map  $\varphi_m$  is birational for all sufficiently large integer  $m$ . It is natural and interesting to find an effective bound for  $m$ . By the result of Hacon-McKernan [7], Takayama [10] and Tsuji (cf. [11]), one knows that there exists a positive integer  $r_n$  depending only on  $n = \dim(V)$  such that  $\varphi_m$  is birational for all  $m \geq r_n$ . In the case of threefolds, the previous work of the first two authors (cf. [3, 4]) shows that  $r_3 \leq 73$ .

In this note we study irregular threefolds (i.e.  $q(V) > 0$ ) of general type. Recent developments on the technique inspired by the Fourier-Mukai transform show that the geometry of irregular threefolds is very similar to that of general fibers of the Albanese map. Noting that the 5-canonical map of a general type surface is birational, one may expect that  $\varphi_5$  is birational too for those threefolds which admit a fibration over (a subvariety of) an abelian variety. Indeed, given a nonsingular projective irregular threefold of general type, it has been proved by Chen and Hacon [5, Theorem 2.8, Proposition 2.9] that  $\varphi_m$  is birational for all  $m \geq 7$  and, moreover, that  $\varphi_5$  is birational if  $\chi(\omega_X) > 0$ .

The aim of this paper is to prove the following:

**Theorem 1.1.** *Let  $V$  be a nonsingular projective irregular 3-fold of general type. Then  $\varphi_6$  is birational.*

## 2. Proof of the main theorem

**2.1. Reductions.** In order to prove Theorem 1.1, we have the following reduction to special cases:

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- (1) Let  $V$  be a nonsingular projective 3-fold of general type. Take any birational projective model  $W$  of  $V$  so that  $W$  has at worst canonical singularities. Then  $V$  and  $W$  share the same birational invariants and  $\Phi_{mK_W} \approx \Phi_{mK_V}$ . Therefore it is sufficient to prove the statement of Theorem 1.1 just replacing  $V$  with any suitable birational model  $W$ .
- (2) By Chen and Hacon [5, Proposition 2.9], one only needs to consider the following situation (since, otherwise,  $|6K|$  gives a birational map):
  - (†) The Albanese map of  $V$  induces the fibration  $a_V : V \rightarrow C$  onto an elliptic curve  $C$ , of which the general fiber is a  $(1, 2)$  surface  $S$ , i.e.  $(K_{S_0}^2, p_g(S)) = (1, 2)$ , where  $S_0$  is assumed to be the minimal model of  $S$ .
- (3) Also due to Chen and Hacon [5, Theorem 1.1], we may assume that  $\chi(\mathcal{O}_V) \geq 0$  (since, otherwise,  $|5K|$  gives a birational map).
- (4) By running the minimal model program, one gets a relative minimal model  $X \rightarrow C$  of  $a_V$  where  $X$  has  $\mathbb{Q}$ -factorial terminal singularities. Then  $K_{X/C}$  is nef (see, for instance, Ohno [8, Theorem 1.4]), which means that  $X$  is minimal since  $K_C$  is trivial. In the proof of Theorem 1.1, we may and do replace  $V$  by a minimal model  $X$  (i.e.  $K_X$  nef) which has at worst  $\mathbb{Q}$ -factorial terminal singularities.

**Corollary 2.1.** *Suppose  $V$  (or  $X$ ) satisfies 2.1(2) and 2.1(3). Then  $q(X) = 1$ ,  $p_g(X) = h^2(\mathcal{O}_X) \leq 2$  and thus  $\chi(\mathcal{O}_X) = 0$ .*

*Proof.* Clearly one has  $q(V) = 1$ . Since  $q(S) = 0$ , we see  $h^2(\mathcal{O}_V) = h^1(a_*\omega_V)$ . So one has  $\chi(\mathcal{O}_V) = h^2(\mathcal{O}_V) - p_g(V) = h^1(a_*\omega_V) - h^0(a_*\omega_V) = -\deg(a_*\omega_{V/C}) \leq 0$  by the semi-positivity theorem of Fujita [6]. Thus  $\chi(\mathcal{O}_V) = 0$  and  $p_g(V) = h^2(\mathcal{O}_V)$ . Also by the semi-positivity of  $a_*\omega_V = a_*\omega_{V/C}$ ,  $p_g(V) = h^2(\mathcal{O}_V) = h^1(a_*\omega_V) \leq \text{rk}(a_*\omega_V) = 2$ . By Reid's R-R formula in [9], one can see  $P_2(V) > 0$  and  $P_{m+1}(V) > P_m(V)$  for all  $m \geq 2$ .  $\square$

**2.2. Definitions and lemmas.** Before proving the main result, we would like to recall some notion and results in Chen and Hacon [5].

**Definition 2.2.** For any vector bundle  $E$  on an elliptic curve, we write  $E = \oplus E_i$ , where each  $E_i$  is indecomposable. We define  $\nu(E) := \min\{\mu(E_i)\}$ , where  $\mu(E_i) = \frac{\deg(E_i)}{\text{rk}(E_i)}$  is the slope of  $E_i$ .

**Definition 2.3.** A coherent sheaf  $\mathcal{F}$  on an abelian variety  $A$  is said to be  $IT^0$  if  $H^i(A, \mathcal{F} \otimes P) = 0$  for all  $i > 0$  and all  $P \in \text{Pic}^0(A)$ .

**Lemma 2.4.** ([5, Lemma 4.8]) *Let  $E_1, E_2$  be vector bundles on an elliptic curve.*

- (1) *If  $E_1, E_2$  are indecomposable and  $\text{Hom}(E_1, E_2) \neq 0$ , then  $\mu(E_2) \geq \mu(E_1)$ .*

(2) If there exists a surjective map  $E_1 \rightarrow E_2$ , then  $\nu(E_2) \geq \nu(E_1)$ .

**Lemma 2.5.** ([5, Lemma 4.10]) *Let  $E$  be an  $IT^0$  vector bundle on an elliptic curve which admits a short exact sequence*

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0$$

*of coherent sheaves such that  $Q$  has generic rank = 0 (resp.  $\leq 1$ ). Then  $\nu(E) \geq \nu(F)$  (resp.  $\nu(E) \geq \min\{1, \nu(F)\}$ ).*

**2.3. Multiplication maps  $\varphi_{m,n}$  and  $\psi_{m,n}$ .** Consider the fibration  $a : X \rightarrow C$  as in 2.2. Let  $F$  be a general fiber  $F$  of  $a$ . Let  $R_m := H^0(F, \omega_F^m)$  and  $E_m := a_*\omega_X^m$ . By Chen and Hacon [5, Lemma 4.1],  $E_m$  is an  $IT^0$  vector bundle of rank  $P_m(F)$  for all  $m \geq 2$ . We also remark that  $\nu(E_m) \geq 0$  by the semi-positivity theorem (see, for instance, Viehweg [12]) and Atiyah's description of vector bundles over elliptic curves (cf. [1]). We consider the multiplication map of pluricanonical systems on the fiber  $F$ , say

$$\varphi_{m,n} : R_m \otimes R_n \rightarrow R_{m+n}.$$

This naturally induces a map between vector bundles

$$\psi_{m,n} : E_m \otimes E_n \rightarrow E_{m+n}$$

where  $m, n > 0$ . Clearly if cokernel of  $\varphi_{m,n}$  has dimension  $\leq r$ , then cokernel of  $\psi_{m,n}$  has rank  $\leq r$ .

**2.4. Proof of Theorem 1.1.** First of all, we recall that the linear system  $|6K_V|$  separates two general points on two distinct general fibers of the Albanese map  $a_V$  (see [5, Theorem 2.8 (2)]). Hence we just need to show that  $|6K_V|$  separates two general points on a general fiber of  $a_V$  to conclude the proof of Theorem 1.1.

We now take the birational model  $a : X \rightarrow C$  of  $V$  as in 2.1(1)~(4).

*Step 1.* We construct a relative canonical model  $W \rightarrow C$  of  $a$ .

We may take an integer  $m \gg 0$  and pick a very ample divisor  $L$  on  $C$  so that

- i. for the general fiber  $F$  of  $a$ ,  $|mK_F|$  is base point free and  $\Phi_{|mK_F|}(F)$  is the canonical model of  $F$ ;
- ii.  $|a^*L + mK_X|$  is free;
- iii.  $a_*\omega_X^m \otimes \mathcal{O}_C(L)$  is generated by global sections and then the restriction map  $H^0(X, a^*L + mK_X) \rightarrow H^0(F, mK_F)$  is surjective for general  $F$ ;
- iv.  $a_*\omega_X^2 \otimes \mathcal{O}_C(L)$  is generated by global sections and then the restriction map  $H^0(X, a^*L + 2K_X) \rightarrow H^0(F, 2K_F)$  is surjective for general  $F$ .

The linear system  $|a^*L + mK_X|$  defines a morphism  $X \rightarrow \mathbb{P}^N$  over  $C$  and let  $W$  be its image. Then we get a relative canonical model  $g : W \rightarrow C$ . Clearly, by definition,  $a$  factors through  $g$ . Denote by  $G$

the general fiber of  $g$ . Then  $W|_G$  is exactly the canonical model of  $F$  for general  $F$ .

*Step 2.* The relative bicanonical map  $h : Y \rightarrow C$  of  $g$ .

It is known (cf. Catanese [2, 1.3 Example]) that the canonical model  $G$  of any  $(1,2)$  surface is a degree 10 weighted hypersurface, with at worst rational double points, in  $\mathbb{P}(1,1,2,5)$ . Namely, if  $x, y, z, u$  are coordinates of  $\mathbb{P}(1,1,2,5)$ , then  $G$  is given by the homogeneous equation  $u^2 - f_{10}(x, y, z)$  for some homogeneous polynomial  $f_{10}(x, y, z)$  of degree 10 in  $x, y, z$ . Furthermore the bicanonical map  $\varphi_2$  of  $G$  is a double covering onto  $\mathbb{P}(1,1,2)$  branched along a reduced divisor  $B_0 = \text{div}(f_{10}) \subset \mathbb{P}(1,1,2)$  of degree 10.

By the choice of  $m$ , we may assume that the rational map

$$\Phi_{|a^*(L)+2K_X|} : X \dashrightarrow Y$$

factors through  $W$  where  $Y$  is assumed to be the closure of the image. Notice also that  $a : X \rightarrow C$  factors through  $Y$ . Moreover, there is a natural injection  $Y \hookrightarrow \mathbb{P}(a_*\omega^2) = \mathbb{P}(E_2)$ , where  $\mathbb{P}(E_2)$  is a  $\mathbb{P}^3$ -bundle over  $C$ . We have a new fibration  $h : Y \rightarrow C$  which is induced from the bicanonical map of  $g$ .

Let  $H$  be the general fiber of  $h : Y \rightarrow C$ . Over a general point of  $C$ , we have morphisms  $F \rightarrow G \rightarrow H$  where  $F$  is a minimal  $(1,2)$  surface,  $G$  is the degree 10 hypersurface in  $\mathbb{P}(1,1,2,5)$  with RDPs and  $H \cong \mathbb{P}(1,1,2)$ . We have seen that both  $X \dashrightarrow Y$  and  $W \dashrightarrow Y$  are well-defined over general points of  $C$ . Replacing both  $X$  and  $W$  with suitable birational models  $\hat{X}$  and  $\hat{W}$  by a necessary birational modification to those indeterminacies, we have the following commutative diagram:

$$\begin{array}{ccccccc} \hat{X} & \xrightarrow{\sigma} & \hat{W} & \xrightarrow{\tau} & Y & \longrightarrow & \mathbb{P}(E_2) \\ \hat{a} \downarrow & & \hat{g} \downarrow & & h \downarrow & & \downarrow p \\ C & \xrightarrow{=} & C & \xrightarrow{=} & C & \xrightarrow{=} & C. \end{array}$$

where  $\hat{X}$  (resp.  $\hat{W}$ ) coincides with  $X$  (resp.  $W$ ) over a Zariski open subset  $U$  of  $C$  and  $\hat{a}$  (resp.  $\hat{g}$ ) factors through  $a$  (resp.  $g$ ).

*Step 3.* The decomposition of  $E_m$  by the double covering construction.

Shrinking  $U$ , if necessary, so that  $\tau : W_U = \hat{W}_U \rightarrow Y_U$  is a double covering branched along an even reduced divisor  $B_U \subset Y_U$ . Let  $B_1$  be the closure of  $B_U$  in  $Y$ . Then

$$\mathcal{O}_Y(B_1) = \mathcal{O}_{\mathbb{P}(E_2)}(10) \otimes p^*\mathcal{M}|_Y$$

for some line bundle  $\mathcal{M}$  on  $C$ . Set  $B = B_1$  (resp.  $B = B_1 + H_0$ ) if  $\deg(\mathcal{M})$  is even (resp. odd), then  $\mathcal{O}_Y(B) = \mathcal{L}^{\otimes 2}$ , where  $\mathcal{L} = (\mathcal{O}_{\mathbb{P}(E_2)}(5) \otimes \pi^*\mathcal{M}')|_Y$  for some  $\mathcal{M}'$ .

Let  $\mu : \tilde{Y} \rightarrow Y$  be the log resolution of  $(Y, B)$  and let  $\tilde{B} := \mu^* B - 2\lfloor \frac{\mu^* B}{2} \rfloor$  and  $\tilde{\mathcal{L}} = \mu^* \mathcal{L} \otimes \mathcal{O}(-\lfloor \frac{\mu^* B}{2} \rfloor)$ . Clearly  $\tilde{B}$  is a reduced *SNC* divisor and  $\mathcal{O}(\tilde{B}) = \tilde{\mathcal{L}}^{\otimes 2}$ . Let  $\tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$  be the double cover over  $\tilde{Y}$  branched along  $\tilde{B}$ . One sees that  $\tilde{X}$  has at worst canonical singularities by local consideration. We thus have

$$\tilde{\pi}_* \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}) = \mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^m \oplus \mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^{m-1}$$

for all  $m > 0$ . Now if we take a common birational modification to both  $\hat{X}$  and  $\tilde{X}$  and take push-forwards in two directions respectively, we shall get the following decomposition

$$E_m := E_{m,0} \oplus E_{m,1},$$

where

$$\begin{aligned} E_m &:= a_* \mathcal{O}_X(mK_X); \\ E_{m,0} &:= h_* \mu_* (\mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^m); \\ E_{m,1} &:= h_* \mu_* (\mathcal{O}_{\tilde{Y}}(mK_{\tilde{Y}}) \otimes \tilde{\mathcal{L}}^{m-1}). \end{aligned}$$

*Step 4.* Calculating  $\nu(E_{6,i})$ .

It is rather easy to check that  $\text{rk}(E_{m,0}) = h^0(H, \mathcal{O}(m))$  and  $\text{rk}(E_{m,1}) = h^0(H, \mathcal{O}(m-5))$  for a general fiber  $H$  of  $h$ . Indeed, for  $t \in U$ ,

$$\begin{aligned} E_m \otimes k(t) &\cong H^0(F_t, \mathcal{O}_{F_t}(mK_X)) \cong H^0(G_t, \mathcal{O}_{G_t}(mK_W)) \\ &\cong H^0(\mathbb{P}(1, 1, 2, 5), \mathcal{O}(m)), \end{aligned}$$

$$\begin{aligned} E_{m,0} \otimes k(t) &\cong H^0(H_t, \mathcal{O}_{H_t}(mK_Y + mL)) \cong H^0(\mathbb{P}(1, 1, 2), \mathcal{O}(m)) \text{ and} \\ E_{m,1} \otimes k(t) &\cong H^0(H_t, \mathcal{O}_{H_t}(mK_Y + (m-1)L)) \cong H^0(\mathbb{P}(1, 1, 2), \mathcal{O}(m-5)). \end{aligned}$$

It follows that  $\psi_{m,n}$  induces a map

$$E_{m,0} \otimes E_{n,0} \rightarrow E_{m+n,0}.$$

Since  $E_{m,0} = E_m$  for  $m \leq 4$ . One sees that

$$\psi_{4,2} : E_4 \otimes E_2 \cong E_{4,0} \otimes E_{2,0} \rightarrow E_{6,0}$$

is generically surjective. Since  $E_2$  is a non-zero  $IT^0$  sheaf, we have  $h^0(E_2) \geq 1$ . Hence  $\nu(E_2) \geq \frac{1}{4}$ . Since  $\psi_{2,2}$  is generically surjective, we have  $\nu(E_4) \geq \nu(E_{4,0}) \geq \frac{1}{2}$  by Lemma 2.5. Similarly,  $\nu(E_{6,0}) \geq \frac{3}{4}$ . Moreover,  $E_{6,1}$  is  $IT^0$  of rank 2 by Lemma 2.4, hence  $\nu(E_{6,1}) \geq \frac{1}{2}$ .

*Step 5.* Birationality of  $\varphi_6$ .

We need the following:

**Lemma 2.6.** *Let  $\mathcal{F}$  be a coherent sheaf on  $X$  and  $\mathcal{E} := a_* \mathcal{F}$  on  $C$ . Suppose that  $\mathcal{E}$  is an  $IT^0$  vector bundle. Then for any general fiber  $X_t$ , the image of the restriction map  $H^0(C, \mathcal{E}) \cong H^0(X, \mathcal{F}) \xrightarrow{\text{res}} H^0(X_t, \mathcal{F}|_{X_t})$  has dimension  $\geq \text{rk}(\mathcal{E}) \cdot \min\{\nu(\mathcal{E}), 1\}$ .*

*Proof.* Take the decomposition of  $\mathcal{E} = \oplus \mathcal{E}_i$  into indecomposable bundles. For each  $i$ , there is an induced exact sequence

$$0 \rightarrow \mathcal{E}_i \otimes \mathcal{O}_C(-t) \rightarrow \mathcal{E}_i \rightarrow \mathcal{E}_i \otimes k(t) \rightarrow 0.$$

Let  $d_i = \deg(\mathcal{E}_i)$  and  $r_i = \text{rk}(\mathcal{E}_i)$ , then  $\mathcal{E}_i \otimes \mathcal{O}_C(-t)$  has rank  $r_i$  and degree  $d_i - r_i$ . If  $d_i = r_i$ , then  $\mathcal{E}_i \otimes \mathcal{O}_C(-t)$  is a indecomposable rank  $r_i$  vector bundle of degree 0. Hence  $\mathcal{E}_i \otimes \mathcal{O}_C(-t) \cong U_{r_i} \otimes P$  for some  $P \in \text{Pic}^0(C)$  and  $U_{r_i}$  is a unipotent vector bundle (cf. [1]). Whenever  $P = \mathcal{O}$ , we pick  $t' \neq t$  and consider  $\mathcal{E}_i \otimes \mathcal{O}_C(-t') \cong U_{r_i} \otimes \mathcal{O}(-t' + t)$  instead so that it has no global section. Hence we may and do assume that  $H^0(\mathcal{E}_i \otimes \mathcal{O}_C(-t)) = 0$  for general  $t \in C$  if  $d_i = r_i$ .

It now follows that  $h^0(\mathcal{E}_i \otimes \mathcal{O}_C(-t)) = \max\{0, d_i - r_i\}$  for general  $t$ . Hence the image of  $H^0(\mathcal{E}_i) \rightarrow H^0(\mathcal{E}_i \otimes k(t))$  has dimension  $d_i$  (resp.  $r_i$ ) if  $d_i < r_i$  (resp.  $d_i \geq r_i$ ). The statement now follows by simply taking the sum.  $\square$

Let  $V_{m,i}$  ( $i = 0, 1$ ) be the image of the following map

$$H^0(C, E_{m,i}) \hookrightarrow H^0(C, E_m) \xrightarrow{\text{res}} H^0(F_t, \mathcal{O}(mK)|_{F_t})$$

for a general point  $t \in C$ . Then we have  $\dim V_{6,0} \geq 12$  and  $\dim V_{6,1} \geq 1$  by Lemma 2.6.

**Claim.** *The subsystem given by the vector space*

$$V_{6,0} + V_{6,1} \subset H^0(G_t, \mathcal{O}(6))$$

*gives a birational map on  $G_t$  for all general  $t \in C$ .*

We consider the local sections explicitly. Let  $x, y, z, u$  be all the 4 coordinates of  $\mathbb{P}(1, 1, 2, 5)$  with weights 1, 1, 2, 5. Then  $E_{m,0} \otimes k(t)$  is generated by sections in  $\{x^i y^j z^k | i + j + 2k = m\}$  and  $E_{m,1} \otimes k(t)$  is generated by sections in  $\{x^i y^j z^k u | i + j + 2k = m - 5\}$ . In a word, either  $xu$  or  $yu$  extends to global sections in  $H^0(X, 6K_X)$ . Furthermore, at least 12 linearly independent sections in  $E_{m,0} \otimes k(t)$  can be extended to global sections in  $H^0(X, 6K_X)$ .

To prove the claim, we put  $H = H_t$  and let  $\Sigma_0 \subset H^0(H, \mathcal{O}_H(6))$  (resp.  $\Sigma_1 \subset H^0(H, \mathcal{O}_H(6))$ ) be the subspace spanned by  $\{x^6, \dots, y^6\}$  (resp. by  $\{x^4 z, x^3 y z, \dots, y^4 z\}$ ). We see that  $\dim \Sigma_0 = 7$  and  $\dim \Sigma_1 = 5$ . By dimensional considerations, one has  $\dim V_{6,0} \cap \Sigma_0 \geq 3$  and  $\dim V_{6,0} \cap \Sigma_1 \geq 1$ . Pick linearly independent elements  $\sigma_{0,1}, \sigma_{0,2}, \sigma_{0,3} \in V_{6,0} \cap \Sigma_0$  and  $z\sigma_1 \in V_{6,0} \cap \Sigma_1$ . We consider the map  $\tilde{\varphi} : H \dashrightarrow \mathbb{P}^3$  defined by these 4 sections. It is easy to see that  $\tilde{\varphi}$  has image of dimension 2. Indeed, consider the map  $\varphi : H \dashrightarrow \mathbb{P}^{11}$  given by  $V_{6,0}$  with image  $H'$ . Since  $\varphi$  factors through  $\tilde{\varphi}$ , one sees that  $H'$  is a surface and clearly  $\deg(H') \geq 10$ . Since

$$\deg(\varphi) \cdot \deg(H') \leq (\mathcal{O}_H(6) \cdot \mathcal{O}_H(6))_H = 18,$$

it follows that  $\varphi$  has degree 1, hence is birational.

Since  $G_t \cong X_{10} \rightarrow \mathbb{P}(1, 1, 2) \cong H$  is a  $2 : 1$  map and  $u$  can separate points on general fibers of this double covering. Hence the sections in  $V_{6,1}$  separate points on general fibers of this double covering.

The Claim now follows and hence this completes the proof of Theorem 1.1.  $\square$

**Example 2.7.** Suppose that there exists a minimal irregular threefold  $X$  with a fibration  $f : X \rightarrow C$  fibered by  $(1, 2)$  surfaces. Suppose that  $K_X^3 = \frac{1}{2}$  and  $B(X) = \{3 \times (1, 2)\}$ . By Reid's R-R formula, one has  $P_2(X) = 1$ ,  $P_3(X) = 2$ ,  $P_4(X) = 5$ ,  $P_5(X) = 9$  and  $P_6(X) = 16$ . We show that  $|5K_X|$  may be non-birational.

Note that  $\text{rk}(E_{5,0}) = 12$  and  $\text{rk}(E_{5,1}) = 1$ . Assume  $h^0(E_{5,0}) = 8$  and  $h^0(E_{5,1}) = 1$ .

Now  $H^0(F_t, 5K|_{F_t})$  is generated by

$$\{x^5, \dots, y^5, x^3z, x^2yz, xy^2z, y^3z, xz^2, yz^2, u\}.$$

If  $V_{5,1}$  is generated by  $\{x^5, \dots, y^5, xz^2, yz^2\}$  and  $V_{5,2}$  is generated by  $u$ , then these sections can not distinguish points  $(x_0, y_0, z_0, u_0)$  from  $(x_0, y_0, -z_0, u_0)$ . In other words, it may only give a  $2 : 1$  map on  $F_t$  instead of a birational map.

However, we do not know whether this kind of examples really exists or not.

## REFERENCES

- [1] M. Atiyah, *Vector bundles over an elliptic curves*, Proc. London Math. Soc. **7**(1957), 414-452.
- [2] F. Catanese, *Canonical rings and "special" surfaces of general type*. Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 175-194, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [3] J. A. Chen, M. Chen, *Explicit birational geometry of threefolds of general type, I*, Ann. Sci. Ecole Norm. Sup. (**43**) 2010, 365-394
- [4] J. A. Chen, M. Chen, *Explicit birational geometry of threefolds of general type, II*, **86** (2010), 237-271
- [5] J. A. Chen, C. D. Hacon, *Pluricanonical systems on irregular 3-folds of general type*. Math. Z. **255**(2007), no. 2, 343-355
- [6] T. Fujita, *On Kaehler fiber spaces over curves*. J. Math. Soc. Japan **30** (1978), no. 4, 779-794
- [7] C. D. Hacon and J. McKernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), 1-25.
- [8] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*. J. Math. Soc. Japan **44** (1992), no. 4, 643-666.
- [9] M. Reid, *Young person's guide to canonical singularities*, Proc. Symposia in pure Math. **46**(1987), 345-414.
- [10] S. Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. **165** (2006), 551 - 587.
- [11] H. Tsuji, *Pluricanonical systems of projective varieties of general type, I*, Osaka J. Math. **43** (2006), 967-995.

- [12] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*. Proc. Algebraic Varieties and Analytic Varieties, Tokyo 1981. Adv. Studies in Math. **1**, Kinokunya-North-Holland Publ. 1983, 329-353

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